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Corresponding Author: Prof. Rossana Vermiglio,

Corresponding Author's Institution: University of Udine

First Author: Rossana Vermiglio

Order of Authors: Rossana Vermiglio

Abstract: Abstract Delay Differential Equations (ADDEs) extend Delay Differential Equations (DDEs) from finite to infinite dimension. They arise in many application fields. From a dynamical system point of view, the stability analysis of a steady-state solution is the first relevant question, which can be reduced to the stability of the zero solution of the corresponding linearized system. In the understanding of the linear case, the essential and the non-essential spectra of the infinitesimal generator are crucial. We propose to extend the infinitesimal generator approach developed for linear DDEs to approximate the non-essential spectrum of linear ADDEs. We complete the paper with the numerical results for a homogeneous neural field model with transmission delay of a single population of neurons.

Numerical approximation of the non-essential spectrum of abstract delay differential equations

Rossana Vermiglio^a

^a*Department of Mathematics and Computer Science, University of Udine (Italy)*

Abstract

Abstract Delay Differential Equations (ADDEs) extend Delay Differential Equations (DDEs) from finite to infinite dimension. They arise in many application fields. From a dynamical system point of view, the stability analysis of a steady-state solution is the first relevant question, which can be reduced to the stability of the zero solution of the corresponding linearized system. In the understanding of the linear case, the essential and the non-essential spectra of the infinitesimal generator are crucial. We propose to extend the infinitesimal generator approach developed for linear DDEs to approximate the non-essential spectrum of linear ADDEs. We complete the paper with the numerical results for a homogeneous neural field model with transmission delay of a single population of neurons.

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1. Introduction

Abstract Delay Differential Equations (ADDEs) extend Delay Differential Equations (DDEs) from finite to infinite dimension. They arise in different application fields [1, 2, 10, 13, 16]. In this paper our interest is in the stability of steady-state solutions of ADDEs. The principle of linearized stability allows one to turn the analysis of stability of a steady-state solution into the stability of the zero solution of the corresponding linearized system. Therefore the understanding of the linear case plays a crucial role in the analysis of the asymptotic behaviour, which can be analyzed by the spectrum of the

Email address: rossana.vermiglio@uniud.it (Rossana Vermiglio)

solution operator semigroup and of the associated Infinitesimal Generator (IG) operator. Dealing with linear DDEs in finite-dimensional spaces, the semigroup is eventually compact and the IG has only point spectrum [8]. Therefore the stability of the zero solution is characterized by the position in the complex plane of the eigenvalues of the generator: the zero solution is asymptotically stable if and only if the real part of the rightmost eigenvalue is negative. Recently a numerical approach to approximate the eigenvalues of the infinitesimal generator associated to DDEs has been proposed [3, 4, 7]. By using the pseudospectral discretization technique, the original infinite-dimensional eigenvalue problem is turned into a matrix eigenvalue problem. The spectral accuracy of the so-called IG-approach allows to obtain very accurate approximations with a small discretization parameter. It has been extended to linear partial differential equations of evolution involving time delay, coupling the pseudospectral method with the spectral method [6]. Partial Retarded Differential Equations (PRFDEs) have been deeply studied in [16]. The linear PRDEs can be recasted as linear ADDEs, whose particular structure allows to prove that the semigroup is eventually compact and the associated infinitesimal generator has only point spectrum. But when dealing with general linear ADDEs, the eventually compactness of the solution semigroup is generally lacking and the essential and non-essential spectrum play a relevant role in the stability analysis. Having in mind the IG-approach for DDEs and for PRFDEs, we investigate how we can extend it to construct an approximation of the non-essential spectrum of IG for linear ADDEs. For the neural field models with space-dependent delay introduced in [10, 14] and analyzed in [13] the essential spectrum consists of a single point in the left-half complex plane and as a consequence the stability of the zero-solution may be inferred from the location of the non-essential spectrum in the complex plane. For this reason we propose the model as test equation. But it is not the only situations one can think of and other examples of ADDEs can be found in the literature [1, 2, 16]. As further example we recall the reformulation of DDEs with uncertain parameters as ADDEs considered in [15] and presented at conference “SDS 2014 - Structural Dynamical Systems: Computational Aspects”.

The paper is organized as follows. In Section 2 we introduce the notations and we summarize some general results on the well-posedness of the initial value problems for nonlinear autonomous ADDEs. Moreover we recall the linearization principle and the semigroup approach for the stability analysis of the zero-solution of linear ADDEs, primarily following [1]. In Section 3 the

IG-approach is applied to approximate the non-essential spectrum of linear ADDEs. Finally the numerical results are presented in Section 4.

2. Abstract delay differential equations

Let \mathcal{Y} be a infinite-dimensional Banach space and denote $|\cdot|_{\mathcal{Y}}$ the norm. Given the delay $\tau > 0$, we consider the Banach space $\mathcal{C} := C([-\tau, 0]; \mathcal{Y})$ of \mathcal{Y} -valued continuous functions defined on the delay interval $[-\tau, 0]$ equipped with the norm $\|\Phi\| = \max_{\theta \in [-\tau, 0]} |\Phi(\theta)|_{\mathcal{Y}}$.

Given a continuous function $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{Y}$, we define *Abstract Delay Differential Equation* (ADDE) the following relation

$$\frac{dY(t)}{dt} = \mathcal{F}(Y_t), \quad t \geq t_0, \quad (1)$$

where $Y_t \in \mathcal{C}$, defined as

$$Y_t(\theta) = Y(t + \theta), \quad \theta \in [-\tau, 0],$$

represents the *state* at time t and \mathcal{C} is the *state-space*.

Such equations arise in many different application fields. In fact several classes of differential equations, such as delayed reaction-diffusion equations, wave equations and age-dependent populations equations can be reformulated as ADDEs [2, 16]. Others examples can be found in [1] and in the references therein. Recently [10, 14] consider neural field models with space-dependent delay reformulated as non autonomous ADDE on $\mathcal{Y} = L^2(\Omega, \mathbb{R}^d)$, where Ω is an open subset of \mathbb{R}^q , $q = 1, 2, 3$. In particular in [10] the existence and the uniqueness of the solution given an initial condition is shown, while in [14] the focus is on the center manifold theorem. For such models, similar questions are analyzed by sun-star calculus on $\mathcal{Y} = C(\overline{\Omega})$ in [13], where one finds also some remarks about the different choices of the space \mathcal{Y} . In what follows we briefly recall some basic results and we referee to [1, 2, 8] for further details.

Given $\Phi \in \mathcal{C}$, an initial value problem (IVP) for (1) is

$$\begin{cases} \frac{dY(t)}{dt} = \mathcal{F}(Y_t), & t \geq t_0, \\ Y_{t_0} = \Phi. \end{cases} \quad (2)$$

A function Y is a *solution* of the IVP (2) on $([t_0 - \tau, t_0 + \alpha])$ if there exists $\alpha > 0$ such that $Y \in C([t_0 - \tau, t_0 + \alpha]; \mathcal{Y}) \cap C^1([t_0, t_0 + \alpha]; \mathcal{Y})$ and it satisfies (2) on $([t_0 - \tau, t_0 + \alpha])$. If $\alpha = +\infty$ we say that Y is a *global solution*.

Here we recall an existence and uniqueness result which requires a global Lipschitz condition for the function \mathcal{F} . In the literature one can find local existence and uniqueness results, relaxing this condition.

Theorem 1. *If $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{Y}$ is a continuous and globally Lipschitz function, i.e.*

$$|\mathcal{F}(\Phi) - \mathcal{F}(\Psi)|_{\mathcal{Y}} \leq L\|\Phi - \Psi\|, \Phi, \Psi \in \mathcal{C},$$

where L is a positive constant, then any $\Phi \in \mathcal{C}$ fixes a unique global solution of the IVP (2).

2.1. The semigroup approach for linear abstract delay differential equations

Our aim is to study the local behaviour of the solutions of the ADDE (1) around a steady-state solution, which is a solution $\bar{Y} \in \mathcal{Y}$ independent of t . Let \mathcal{F} be a continuously differentiable function. The principle of linearized stability also applied to ADDE (1) and then we can reduce the original problem to the stability analysis of the zero-solution of the corresponding linearized ADDE

$$\frac{dY(t)}{dt} = \mathcal{L}Y_t, \quad t \geq 0, \quad (3)$$

where $\mathcal{L} : \mathcal{C} \rightarrow \mathcal{Y}$ is the Fréchet derivative of \mathcal{F} at \bar{Y} , i.e. $\mathcal{L} = D\mathcal{F}(\bar{Y})$. The operator \mathcal{L} is linear and bounded. The Theorem 1 ensures that for any $\Phi \in \mathcal{C}$ the IVP

$$\begin{cases} \frac{dY(t)}{dt} = \mathcal{L}Y_t, & t \geq 0, \\ Y_0 = \Phi. \end{cases} \quad (4)$$

has a unique global solution.

Therefore for any $t \geq 0$, we can define the Solution Operator (SO) $\mathcal{T}(t) : \mathcal{C} \rightarrow \mathcal{C}$ as the linear bounded operator which associates to the initial state Φ the state Y_t , i.e.

$$\mathcal{T}(t)\Phi = Y_t, \quad (5)$$

where Y is the solution of (4). The family $\{\mathcal{T}(t)\}_{t \geq 0}$ is a C_0 -semigroup on the space \mathcal{C} .

The Infinitesimal Generator (IG) $\mathcal{A} : Dom(\mathcal{A}) \subseteq \mathcal{C} \rightarrow \mathcal{C}$ associated to $\{\mathcal{T}(t)\}_{t \geq 0}$ is

$$\begin{cases} \mathcal{A}\Phi = \Phi', & \Phi \in Dom(\mathcal{A}) \\ Dom(\mathcal{A}) = \{\Phi \in \mathcal{C} : \Phi' \in \mathcal{C}, \Phi'(0) = \mathcal{L}\Phi\} \end{cases}. \quad (6)$$

It is a closed densely-defined operator.

The spectra of \mathcal{A} and $\mathcal{T}(t)$ play an important role in the stability of the zero-solution. For the spectral analysis it is necessary to work on Banach space on \mathbb{C} and we implicitly assume that \mathcal{C} , \mathcal{Y} and all the operators involved have been complexified [8]. So far we have proceeded as for DDEs, but now we arrive to the main crucial point: in the infinite dimension the eventual compactness of the semigroup is generally lacking. In this more general situation the notions of essential and non-essential are relevant [1, 8, 11, 12]).

The spectrum $\sigma(\mathcal{A})$ of a generic linear (closed or bounded) operator \mathcal{A} can be subdivided into three disjoint subsets

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_c(\mathcal{A}) \cup \sigma_r(\mathcal{A})$$

where $\sigma_p(\mathcal{A})$ is the *point spectrum* (i.e. the set of $\lambda \in \mathbb{C}$ such that $\lambda I - \mathcal{A}$ is not injective), $\sigma_c(\mathcal{A})$ is the *continuous spectrum* (i.e. the set of $\lambda \in \mathbb{C}$ such that $\lambda I - \mathcal{A}$ is injective and the range of $\lambda I - \mathcal{A}$, denoted by $\mathcal{R}(\lambda I - \mathcal{A})$, is not \mathcal{Y} but it is dense in \mathcal{Y}), $\sigma_r(\mathcal{A})$ is the *residual spectrum* (i.e. the set of $\lambda \in \mathbb{C}$ such that $\lambda I - \mathcal{A}$ is injective and $\mathcal{R}(\lambda I - \mathcal{A})$ is not dense in \mathcal{Y}). There is another way to divide the spectrum. The *essential spectrum* $\sigma_e(\mathcal{A})$ of \mathcal{A} is the set of $\lambda \in \sigma(\mathcal{A})$ such that one of the following holds

- $\mathcal{R}(\lambda I - \mathcal{A})$ is not closed,
- the generalized eigenspace associated to λ , i.e.

$$\mathcal{M}_\lambda(\mathcal{A}) = \bigcup_{q=1}^{\infty} \mathcal{N}((\lambda I - \mathcal{A})^q), \quad (7)$$

where \mathcal{N} denotes the null space, is infinite-dimensional,

- λ is a limit point of $\sigma(\mathcal{A})$.

The complementary set, i.e. $(\sigma \setminus \sigma_e)(\mathcal{A})$, is the *non-essential spectrum*. Moreover we recall that the *spectral bound* $s(\mathcal{A})$ is the constant

$$s(\mathcal{A}) = \sup_{\lambda \in \sigma(\mathcal{A})} \Re(\lambda).$$

The *growth bound* ω_0 of the semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$, is defined as

$$\omega_0 = \inf\{\omega \in \mathbb{R} : \text{there exists } M > 0 \text{ such that } \|\mathcal{T}(t)\| \leq M e^{\omega t}, t \geq 0\}.$$

and it is related to the asymptotic behaviour of the semigroup. For the growth bound ω_0 of a semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$, and the spectral bound $s(\mathcal{A})$ of the associated generator \mathcal{A} one has $s(\mathcal{A}) \leq \omega_0$. Since we have that

$$\begin{aligned} e^{\sigma_p(\mathcal{A})t} &= \sigma_p(\mathcal{T}(t)) \setminus \{0\} \\ e^{(\sigma \setminus \sigma_e)(\mathcal{A})t} &= (\sigma \setminus \sigma_e)(\mathcal{T}(t)) \setminus \{0\} \\ e^{\sigma_e(\mathcal{A})t} &\subset \sigma_e(\mathcal{T}(t)) \end{aligned}$$

we can conclude that, if

- $\Re(\lambda) < 0$, for $\lambda \in (\sigma \setminus \sigma_e)(\mathcal{A})$,
- $\sigma_e(\mathcal{T}(t))$ is contained in the interior of the unit disk in the complex plane,

then the semigroup is exponentially stable.

This result states that for the stability it is important to analyze the non-essential spectrum of \mathcal{A} and the essential spectrum of the semigroup of solution operators $\{\mathcal{T}(t)\}_{t \geq 0}$. The latter is related to the distance of the operator to the set of compact operators and it is not an easy task. For the neural field models studied in [13] the essential spectrum of \mathcal{A} is a single point, which is contained in the left half complex plane, and moreover $s(\mathcal{A}) = \omega_0$. Therefore in this case it is important to analyze the non-essential spectrum of \mathcal{A} . This situation seems to be quite common in ADDEs arising in population dynamics [1].

Theorem 2. *Let $\lambda \in (\sigma \setminus \sigma_e)(\mathcal{A})$. Then $\lambda \in \sigma_p(\mathcal{A})$ and, for some positive q , we have that*

$$\mathcal{C} = \mathcal{N}((\lambda I - \mathcal{A})^q) \oplus \mathcal{R}((\lambda I - \mathcal{A})^q)$$

where $\mathcal{N}((\lambda I - \mathcal{A})^q)$ is the generalized eigenspace $\mathcal{M}_\lambda(\mathcal{A})$ and q is the smallest integer with this property (ascent of λ), and $\dim \mathcal{N}((\lambda I - \mathcal{A})^q) < +\infty$ (geometric multiplicity of λ). Moreover \mathcal{A} restricted to $\mathcal{N}((\lambda I - \mathcal{A})^q)$ is bounded with spectrum $\{\lambda\}$ and the subspaces are invariant under the semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$.

PROOF. See [1, Theorem 2].

Let $\Delta(\lambda) : \mathcal{Y} \rightarrow \mathcal{Y}$ be the linear operator given by

$$\Delta(\lambda) := \lambda I_{\mathcal{Y}} - \mathcal{E}(\lambda), \tag{8}$$

where $I_{\mathcal{Y}}$ is the identity operator of \mathcal{Y} , $\mathcal{E}(\lambda) : \mathcal{Y} \rightarrow \mathcal{Y}$ is the linear bounded operator defined by

$$\mathcal{E}(\lambda)Y = \mathcal{L}(e^{\lambda \cdot} \otimes Y) \quad (9)$$

and $e^{\lambda \cdot} \otimes Y \in \mathcal{C}$ is the function

$$(e^{\lambda \cdot} \otimes Y)(\theta) = e^{\lambda \theta} Y, \quad \theta \in [-\tau, 0].$$

Lemma 3. *Let $\lambda \in (\sigma \setminus \sigma_e)(\mathcal{A})$. Then the linear operator $\Delta(\lambda)$ is a Fredholm operator.*

PROOF. See [1, Lemma 41].

The theory of characteristic values holds for analytic Fredholm operator valued function. Let Ω be an open connected set in \mathbb{C} . We say that $\lambda \in \mathbb{C}$ is a *characteristic value* of $\Delta : \lambda \in \Omega \mapsto \Delta(\lambda)$ if there exists $Y \in \mathcal{Y} \setminus \{0\}$ such that

$$\Delta(\lambda)Y = 0. \quad (10)$$

The non-essential eigenvalues of \mathcal{A} are the characteristic value of the Fredholm operator valued function Δ . Moreover if we denote Σ the set of characteristic values of Δ , we get that for any $\lambda^* \in \Sigma$, there exists $\delta > 0$ such that, for $0 < |\lambda - \lambda^*| < \delta$, we have $\lambda \in \Omega \setminus \Sigma$ and

$$\Delta(\lambda)^{-1} = \sum_{i=-q}^{\infty} (\lambda - \lambda^*)^i \Delta_i, \quad (11)$$

where $q > 0$ is the ascent, Δ_i are linear bounded operators for $i \geq -q$, Δ_0 is a Fredholm operator of index zero and $\Delta_{-1}, \dots, \Delta_{-q}$ are operators of finite rank.

By rewriting the linear operator \mathcal{L} as

$$\mathcal{L}\Phi = \mathcal{L}_0\Phi(0) + \mathcal{L}_1\Phi,$$

where the linear operator $\mathcal{L}_0 : \text{Dom}(\mathcal{L}_0) \subset \mathcal{Y} \rightarrow \mathcal{Y}$ describes the no-delay contribution while $\mathcal{L}_1 : \mathcal{C} \rightarrow \mathcal{Y}$ represents the pure delay operator, we can state a useful lemma which links the essential spectrum of \mathcal{A} to the spectral properties of the linear operator \mathcal{L}_0 . It is based on the following representation of the operator (8)

$$\Delta(\lambda) = \lambda I_{\mathcal{Y}} - \mathcal{L}_0 - \mathcal{E}_1(\lambda),$$

where

$$\mathcal{E}_1(\lambda)Y = \mathcal{L}_1(e^\lambda \otimes Y).$$

We conclude with a useful Lemma to locate the essential spectrum of \mathcal{A} .

Lemma 4. *If the operator $\mathcal{E}_1(\lambda)$ is compact for all $\lambda \notin \sigma(\mathcal{L}_0)$, then $\sigma_e(\mathcal{A}) \subseteq \sigma(\mathcal{L}_0)$.*

PROOF. Let $\lambda \in \sigma(\mathcal{A}) \setminus \sigma(\mathcal{L}_0)$. Thus (8) can be written as

$$\Delta(\lambda) = (\lambda I_{\mathcal{Y}} - \mathcal{L}_0)(I_{\mathcal{Y}} - \mathcal{K}(\lambda)),$$

where $\mathcal{K}(\lambda) := (\lambda I_{\mathcal{Y}} - \mathcal{L}_0)^{-1} \mathcal{E}_1(\lambda)$ is compact. From the theory of compact operators, we have that $(I_{\mathcal{Y}} - \mathcal{K}(\lambda))$ has closed range in \mathcal{Y} . Let $Y \in \overline{\mathcal{R}(\Delta(\lambda))}$ and $(\Delta(\lambda)X_n \rightarrow Y$ for some $X_n \in \mathcal{Y}$. Let $Z_n := (I_{\mathcal{Y}} - \mathcal{K}(\lambda))X_n \in \mathcal{R}(I_{\mathcal{Y}} - \mathcal{K}(\lambda))$. We have that $\{Z_n\}$ converges to $(\lambda I_{\mathcal{Y}} - \mathcal{L}_0)^{-1}Y \in \mathcal{R}(I_{\mathcal{Y}} - \mathcal{K}(\lambda))$ and then $Y \in \mathcal{R}(\Delta(\lambda))$. By [1, Lemma 36] we have that the range of $\lambda - \mathcal{A}$ is closed in \mathcal{C} . By definition of essential spectrum we have that $\lambda \notin \sigma_e(\mathcal{A})$.

Remark 1. The Lemma 4 extends the result in [13, Corollary 18], where $\mathcal{Y} = C(\overline{\Omega})$, $\Omega \subset \mathbb{R}^d$ bounded, and $\mathcal{L}_0 = aI$, with $a \in \mathbb{R}$. It allows one to consider a continuous function $a : \overline{\Omega} \rightarrow \mathbb{C}^{d \times d}$ and the multiplication operator $\mathcal{L}_0 Y = a \cdot Y$, with domain $\text{Dom}(\mathcal{L}_0) = \{Y \in \mathcal{Y} : a \cdot Y \in \mathcal{Y}\}$. We have that $\sigma(\mathcal{L}_0) = a(\overline{\Omega})$. For $\mathcal{Y} = L^2(\Omega)$ with $a : \Omega \rightarrow \mathbb{C}$ a measurable function, we have that $\sigma(\mathcal{L}_0) = \overline{a(\Omega)}$ [9, 2]. For a matrix-valued function a , we have $\sigma(\mathcal{L}_0) = \sigma_p(a(\overline{\Omega}))$ for continuous a , and $\sigma(\mathcal{L}_0) = \overline{\sigma_p(a(\Omega))}$ for the measurable a .

3. The IG-approach for linear abstract delay differential equations

The IG-approach developed for *Partial Retarded Functional Differential Equations* (PRFDEs) in the paper [6] can be adapted to the approximation of the non-essential spectrum of the IG associated to (1), but the analysis of convergence needs to be developed. In fact the paper [6] deals with semi-linear PRFDEs, which can be restated as ADDEs of the following type

$$\frac{dY(t)}{dt} = \mathcal{B}Y(t) + \mathcal{F}(Y_t), \quad t \geq 0,$$

where \mathcal{B} is the infinitesimal generator of a compact C_0 -semigroup of bounded linear operators on a complex Banach space \mathcal{Y} . The hypothesis of compactness on \mathcal{B} is motivated by the Laplacian, i.e. $\mathcal{B} = \Delta$, and the assumption

$\mathcal{B} = 0$ is not admissible, when the dimension of $\dim \mathcal{Y} = \infty$. The theory and the numerical method developed respectively in [2, 16] and [6] can not be directly applied to (1) and this fact gives the motivation for the recent papers [1, 10, 14, 13]. In what follows we carry on the same program in [6].

To discretize the infinitesimal generator (6), we need to combine the pseudospectral approach on $[-\tau, 0]$ with an approximation technique in the Banach space \mathcal{Y} . Here we assume that there exists a sequence of finite-dimensional linear subspaces $\{\mathcal{Y}_P\}_{P=0}^\infty$, such that $\overline{\bigcup_{P=0}^\infty \mathcal{Y}_P}$. Moreover we denote $\pi_P : \mathcal{Y} \rightarrow \mathcal{Y}_P$ the operator, which associates to each elements $Y \in \mathcal{Y}$ its approximation $Y_P = \pi_P(Y)$ in the subspace \mathcal{Y}_P , and $\pi_P^2 = \pi_P$.

Remark 2. Different choices for the subspace \mathcal{Y}_P and for the operator π_P can be considered. For instance, when $\mathcal{Y} = C(K; \mathbb{R})$, with K a compact subspace of \mathbb{R}^d , \mathcal{Y}_P and π_P can be defined respectively as the subspace of multivariate polynomials of degree P and the multivariate interpolating operator. The Stone-Weierstrass Theorem ensures the density property. If \mathcal{Y} is an Hilbert space, given an orthogonal basis, Y can be approximated by truncating its Fourier expansion Y_P and π_P is the projection operator.

Let $\mathcal{C}_{N,P} := \Pi_N([-\tau, 0], \mathcal{Y}_P)$ be the space of the \mathcal{Y}_P -valued N -degree polynomials defined on $[-\tau, 0]$. Each function $\Phi \in \mathcal{C}$ is discretized into the polynomial $\Phi_{N,P} \in \mathcal{C}_{N,P}$, which interpolates at the Chebyshev extremal nodes

$$\Theta_N := \{\theta_j = \frac{\tau}{2}(\cos(\frac{j\pi}{N}) - 1), \quad j = 0, 1, \dots, N\}, \quad (12)$$

the values $\Phi_{i,P} := \pi_P(\Phi(\theta_i)) \in \mathcal{Y}_P$, $i = 0, 1, \dots, N$. Note that $\theta_0 = 0$. The discretization version of the infinitesimal generator (6) is the finite-dimensional operator $\mathcal{A}_{N,P} : \mathcal{C}_{N,P} \rightarrow \mathcal{C}_{N,P}$ such that $\Psi_{N,P} = \mathcal{A}_{N,P}\Phi_{N,P}$ is given by

$$\begin{cases} \Psi_{N,P}(\theta_0) = \pi_P \mathcal{L} \Phi_{N,P}, \\ \Psi_{N,P}(\theta_i) = \Phi'_{N,P}(\theta_i), \quad i = 1, \dots, N. \end{cases} \quad (13)$$

By introducing the Lagrange representation of $\Phi_{N,P}$, i.e.

$$\Phi_{N,P}(\theta) = \sum_{j=0}^N \ell_j(\theta) \Phi_{j,P}, \quad \theta \in [-\tau, 0],$$

where ℓ_j are the Lagrange coefficients relevant to (12), we get from (13) the following relations

$$\begin{cases} \Psi_{0,P} = \sum_{j=0}^N \pi_P \mathcal{L}(\ell_j(\cdot) \otimes \Phi_{j,P}), \\ \Psi_{i,P} = \sum_{j=0}^N \ell'_j(\theta_i) \Phi_{j,P}, \quad i = 1, \dots, N. \end{cases} \quad (14)$$

We observe that in some cases, for instance in presence of distributed delays, it could be necessary to introduce an approximation $\tilde{\mathcal{L}} : \mathcal{C} \rightarrow \mathcal{Y}$ of the linear operator \mathcal{L} .

For $\lambda \in \mathbb{C}$ and $Y \in \mathcal{Y}$, let $p_N(\cdot; \lambda, Y) \in \mathcal{C}$ be the N -degree polynomial satisfying

$$\begin{cases} p'_N(\theta_{N,i}; \lambda, Y) = \lambda p_N(\theta_i; \lambda, Y), \quad i = 1, \dots, N, \\ p_N(0; \lambda, Y) = Y, \end{cases} \quad (15)$$

i.e. $p_N(\cdot; \lambda, Y)$ is the collocation polynomial at the nodes θ_i , $i = 1, \dots, N$, for the initial value problem on the space \mathcal{Y} ,

$$\begin{cases} y'(\theta) = \lambda y(\theta), \quad \theta \in [-r, 0], \\ y(0) = Y \end{cases}$$

which solution is

$$y(\cdot; \lambda, Y) = e^{\lambda \cdot} \otimes Y \in \mathcal{C}.$$

By proceeding as in [6, Lemma 3], one can prove the following lemma.

Lemma 5. *Let B be a bounded subset of \mathbb{C} . There exists a positive integer $N_0 = N_0(B)$ such that, for any $N \geq N_0$, $\lambda \in B$ and $Y \in \mathcal{Y}$, there exists a unique N -degree polynomial $p_N(\cdot; \lambda, Y) \in \mathcal{C}$ satisfying (15) and*

$$\|p_N(\cdot; \lambda, Y) - e^{\lambda \cdot} \otimes Y\| \leq \frac{C_0}{\sqrt{N}} \left(\frac{C_1}{N} \right)^N |Y|_{\mathcal{Y}}$$

holds, where $C_0 = C_0(B)$ and $C_1 = C_1(B)$. Moreover, for an open bounded subset B of \mathbb{C} and $N \geq N_0(B)$, the linear operators $\mathcal{S}_N(\lambda) : \mathcal{Y} \rightarrow \mathcal{C}$ and $\mathcal{S}(\lambda) : \mathcal{Y} \rightarrow \mathcal{C}$ given by

$$\mathcal{S}_N(\lambda) Y = p_N(\cdot; \lambda, Y), \quad \mathcal{S}(\lambda) Y = e^{\lambda \cdot} \otimes Y, \quad Y \in \mathcal{Y},$$

are analytic functions of $\lambda \in B$, and, for any $k = 0, 1, 2, \dots$ and $\tilde{B} \subseteq B$,

$$\sup_{\lambda \in \tilde{B}, N \geq N_0} \left\| \mathcal{S}_N^{(k)}(\lambda) - \mathcal{S}^{(k)}(\lambda) \right\| \leq \frac{C_{0,p}}{\sqrt{N}} \left(\frac{C_{1,p}}{N} \right)^N$$

where $\lambda \mapsto \mathcal{S}_N^{(p)}(\lambda)$ and $\lambda \mapsto \mathcal{S}^{(p)}(\lambda)$ are the k -th derivatives of $\lambda \mapsto \mathcal{S}_N(\lambda)$ and $\lambda \mapsto \mathcal{S}(\lambda)$, respectively, $C_{0,k} = C_{0,k}(\tilde{B})$ and $C_{1,k} = C_{1,k}(\tilde{B})$.

By defining the linear bounded operator $\mathcal{E}_N(\lambda) : \mathcal{Y} \rightarrow \mathcal{Y}$

$$\mathcal{E}_N(\lambda)Y = \mathcal{L}p_N(\cdot; \lambda, Y),$$

which is the discrete counterpart of the operator $\mathcal{E}(\lambda)$ in (9), we have that λ is an eigenvalue of $\mathcal{A}_{N,P}$ if and only if there exists $Y \in \mathcal{Y}_P \setminus \{0\}$ such that

$$\Delta_{N,P}(\lambda)Y = 0, \quad (16)$$

where $\Delta_{N,P}(\lambda) : \mathcal{Y}_P \rightarrow \mathcal{Y}_P$ is the linear operator defined by

$$\Delta_{N,P}(\lambda) = \lambda I - \pi_P \mathcal{E}_N(\lambda) |_{\mathcal{Y}_P}. \quad (17)$$

By Lemma 5, we obtain that $\mathcal{E}_N(\lambda)$ and $\mathcal{E}(\lambda)$ are analytic functions of $\lambda \in B$ and, for any $k = 0, 1, 2, \dots$ and $\tilde{B} \subseteq B$,

$$\sup_{\lambda \in \tilde{B}, N \geq N_0} \left\| \mathcal{E}_N^{(k)}(\lambda) - \mathcal{E}^{(k)}(\lambda) \right\| \leq \frac{C_{0,k} \|\mathcal{L}\|}{\sqrt{N}} \left(\frac{C_{1,k}}{N} \right)^N \quad (18)$$

where $\lambda \mapsto \mathcal{E}_N^{(k)}(\lambda)$ and $\lambda \mapsto \mathcal{E}^{(k)}(\lambda)$ are the k -th derivatives of $\lambda \mapsto \mathcal{E}_N(\lambda)$ and $\lambda \mapsto \mathcal{E}(\lambda)$, respectively, and $C_{0,k} = C_{0,k}(\tilde{B})$ and $C_{1,k} = C_{1,k}(\tilde{B})$.

Lemma 6. *Let \mathcal{U} be an open connected set in \mathbb{C} . The operator valued functions $\Delta_{N,P} |_{\mathcal{Y}_P} : \lambda \in \mathcal{U} \mapsto \Delta_{N,P} |_{\mathcal{Y}_P}$ and $\Delta_{N,P} : \lambda \in \mathcal{U} \mapsto \Delta_{N,P}(\lambda)$ have the same characteristic values with the same geometric and partial multiplicities.*

PROOF. The thesis follows by proceeding as in [6, Lemma 4].

The eigenvalues of $\mathcal{A}_{N,P}$ in \mathcal{U} are the characteristic values of $\Delta_{N,P}$. Moreover the algebraic multiplicity as characteristic values of $\Delta_{N,P}$ is the same of the zeros of the equation

$$\det \Delta_{N,P}(\lambda) = 0. \quad (19)$$

The next theorem shows how the eigenvalues of $\mathcal{A}_{N,P}$ approximate as $N, P \rightarrow \infty$ the non-essential spectrum \mathcal{A} , which are the eigenvalues of finite type.

Theorem 7. *Let $\lambda^* \in (\sigma \setminus \sigma_e)(\mathcal{A})$ in the open bounded connected subset \mathcal{U} of \mathbb{C} . Let m and q be the algebraic multiplicity and ascent, respectively, of λ^* as characteristic value of the analytic operator valued function Δ . For any $c > 1$, there exist a neighborhood \mathcal{Y} of λ and positive integer $\overline{N} \geq N_0$, $N_0 = N_0(\mathcal{U})$ given in Lemma 5 such that, for $N \geq \overline{N}$ \mathcal{A} has in \mathcal{Y} the eigenvalues $\lambda_1, \dots, \lambda_\mu$ and the sum of the algebraic multiplicities of $\lambda_1, \dots, \lambda_\mu$, as zeros of (19), is equal to m . Moreover,*

$$\max_{i=1, \dots, \mu} |\lambda_i - \lambda^*| \leq (c\varepsilon_{N,P})^{\frac{1}{q}} \quad (20)$$

holds, where

$$\varepsilon_{N,P} := \max_{\substack{j = -q, \dots, -1, \\ i = j, \dots, -1}} \left\| \left[\pi_P E_N^{(i-j)}(\lambda^*) - E^{(i-j)}(\lambda^*) \right] \Delta_j \right\|,$$

Δ_{-i} , $i = -q, \dots, -1$, are the finite rank operators in the Laurent series (11) of $\Delta^{-1}(\lambda)$ around the characteristic value λ^* , and $E^{(i-j)}$ and $E_N^{(i-j)}$, $j = -q, \dots, i$, are the $(i-j)$ -th derivatives of the analytic operator valued functions E and E_N , respectively.

The error $\varepsilon_{N,P}$ vanishes, as $N, P \rightarrow \infty$, and it can be bounded by

$$\varepsilon_{N,P} \leq \frac{\overline{C}_0}{\sqrt{N}} \left(\frac{\overline{C}_1}{N} \right)^N + \varepsilon_P, \quad (21)$$

where \overline{C}_0 and \overline{C}_1 are constants independent of N and P , and

$$\varepsilon_P := \max_{\substack{j = -q, \dots, -1, \\ i = j, \dots, -1}} \left\| [\pi_P - I] E^{(i-j)}(\lambda^*) \Delta_j \right\|. \quad (22)$$

PROOF. Since Δ is a Fredholm operator by Lemma 3, we can apply the operator version of the Rouché Theorem to state

$$\|[\Delta_{N,P}(\lambda) - \Delta(\lambda)] \Delta(\lambda)^{-1}\| < 1, \quad \lambda \in \Gamma(\rho), \quad (23)$$

where $\rho \in (0, D)$, D is such that

$$|\lambda - \lambda^*| < D \implies \lambda \in \mathcal{U},$$

and

$$\Gamma(\rho) = \{\lambda \in \mathbb{C} : |\lambda - \lambda^*| = \rho\}$$

Now the thesis follows as in [6, Theorem 5].

Note the in case we use an approximation $\tilde{\mathcal{L}}$ of \mathcal{L} , such error has to be added in (21). By computing the rightmost eigenvalue of $\mathcal{A}_{N,P}$ in (13), we have a first insight into the (in)stability properties of the zero solution of linear ADDEs. As already pointed out in the general case a complete analysis needs further investigations, as done for the neural model in [13].

4. Numerical results

To test the numerical approach, we consider the delayed neural field model proposed in [10] and deeply analyzed in [13]. We refer to the latter paper for all the details. In particular we restrict to a single population of neurons, which are distributed on the interval $\bar{\Omega} := [-1, 1]$. Let $V(t, x)$ be the membrane potential at time t , and position x . By taking into account the spatial delay, the potential evolves according to the delay integro-differential equation

$$\frac{\partial V}{\partial t}(t, x) = -\alpha V(t, x) + \int_{\bar{\Omega}} W(x, y) S_{\mu}(V(t - \tau(x, y), y)) dy, \quad t \geq 0, x \in \bar{\Omega}, \quad (24)$$

where $W(x, y) = \bar{c}_1 e^{-\mu_1 |x-y|} + \bar{c}_2 e^{-\mu_2 |x-y|}$, $x, y \in \bar{\Omega}$ is the connectivity kernel, $S_{\mu}(v) = \frac{1}{1+e^{-\mu v}} - \frac{1}{2}$, $v \in \mathbb{R}$ is the activation function and the delay is $\tau(x, y) = \tau_0 + |x - y|$, with $\tau_0 \geq 0$ caused by synaptic processes. By defining $\tau := \max_{x, y \in \bar{\Omega}} \tau(x, y)$ and choosing the Banach space $\mathcal{Y} = C(\bar{\Omega}, \mathbb{R})$, the equation (24) can be modelled as an ADDE (1) on \mathcal{Y} , where $Y(t) := V(t, \cdot)$ and $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{Y}$ is defined as

$$\mathcal{F}(\Phi) = -\alpha \Phi(0) + \int_{\bar{\Omega}} W(\cdot, y) S_{\mu}(\Phi(-\tau(\cdot, y))(y)) dy, \quad \Phi \in \mathcal{C}.$$

The choice of the space \mathcal{Y} is in agreement with [13] and differs from that in [10], where the authors consider $L^2(\Omega, \mathbb{R})$. Since $S_{\mu}(0) = 0$ the model admits the trivial steady state. By linearizing around the zero-solution, we obtain the linear ADDE (3), where the linear operator $\mathcal{L} : \mathcal{C} \rightarrow \mathcal{Y}$ is given by

$$\mathcal{L}\Phi = -\alpha \Phi(0) + \frac{\mu}{4} \int_{\bar{\Omega}} W(\cdot, y) \Phi(-\tau(\cdot, y))(y) dy, \quad \Phi \in \mathcal{C}. \quad (25)$$

To discretize the infinitesimal generator corresponding to (3)-(25), we choose \mathcal{Y}_P as the space of the polynomials of degree P with dimension $d_P = P + 1$.

Any functions $Y \in \mathcal{Y}$ is approximated by the interpolating polynomial Y_P at the $P + 1$ extremal Chebyshev points $x_\beta, \beta = 0, 1, \dots, P$, relevant to $\bar{\Omega}$, and we denote π_P the corresponding interpolation operator. We construct an approximation $\tilde{\mathcal{L}}$ of the operator \mathcal{L} by using a quadrature rule with nodes y_β and weights $b_\beta, \beta = 0, 1, \dots, P'$, i.e.

$$\tilde{\mathcal{L}}\Phi(x) = -\alpha\Phi(0)(x) + \frac{\mu}{4} \sum_{\beta=0}^{P'} b_\beta W(x, y_\beta) \Phi(-\tau(x, y_\beta))(y_\beta), \quad x \in \bar{\Omega}, \Phi \in \mathcal{C}, \quad (26)$$

and finally we obtain

$$\begin{aligned} \pi_P \tilde{\mathcal{L}}\Phi(x) &= -\alpha \sum_{\gamma=0}^P m_\gamma(x) \Phi(0)(x_\gamma) \\ &+ \frac{\mu}{4} \sum_{\beta=0}^{P'} \sum_{\gamma=0}^P b_\beta m_\gamma(x) W(x_\gamma, y_\beta) \Phi(-\tau(x_\gamma, y_\beta))(y_\beta), \quad x \in \bar{\Omega}, \Phi \in \mathcal{C}, \end{aligned} \quad (27)$$

where m_γ are the Lagrange polynomials relevant to the nodes x_γ . Note that (27) involves $d_P \times (P' + 1)$ values of the delay function τ . Hereafter we choose $P' = P$, and the Clenshaw-Curtis quadrature rule, so that $y_\beta = x_\beta, \beta = 0, 1, \dots, P$ and the values $\tau_{\beta\gamma} = \tau(x_\beta, x_\gamma)$ and $w_{\beta\gamma} = W(x_\beta, x_\gamma), \beta, \gamma = 0, 1, \dots, P$ define two symmetric square matrices of dimension d_P . Note in [10, 13] the authors use a uniform grid and the composite trapezoidal rule to approximate the integral in (25). As final step we consider the $N + 1$ extremal Chebyshev points (12), and from (27) we get for the $\mathcal{A}_{N,P} : \mathcal{C}_{N,P} \rightarrow \mathcal{C}_{N,P}$ the following matrix representation

$$\begin{pmatrix} -\alpha I_{d_P} + \frac{\mu}{4} B_0 & \frac{\mu}{4} B_1 & \dots & \frac{\mu}{4} B_N \\ \ell'_0(\theta_1) I_{d_P} & \ell'_2(\theta_1) I_{d_P} & \dots & \ell'_N(\theta_1) I_{d_P} \\ \vdots & \vdots & & \vdots \\ \ell'_0(\theta_N) I_{d_P} & \ell'_1(\theta_N) I_{d_P} & \dots & \ell'_N(\theta_N) I_{d_P} \end{pmatrix},$$

where $B_i := (b_\gamma w_{\beta\gamma} \ell_i(-\tau_{\beta\gamma}))_{\beta, \gamma=0,1,\dots,P} \in \mathbb{R}^{d_P \times d_P}, i = 0, 1, \dots, N$.

By choosing $\tau_0 = 1, \alpha = 1, \bar{c}_1 = -5, \bar{c}_2 = 2, \mu_1 = 2, \mu_2 = 0$ and $\mu = 4$ ([13, Fig.1]), we get the results in Figure 1. Note that the eigenvalues accumulate at -1 which corresponds to the essential spectrum. Moreover the numerical results confirm that convergence is faster for the eigenvalues with smaller modulus and that the spectral accuracy allows to choose

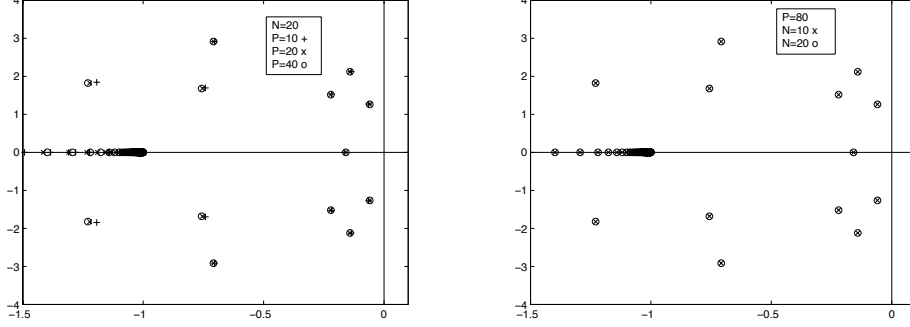


Figure 1: Rightmost part of the spectrum of $\mathcal{A}_{N,P}$ with $P = 10, 20, 40$, $N = 20$ (left) and $N = 10, 20$, $P = 80$ (right)

small N . In Table 1 the errors in the approximation of the real eigenvalue $\lambda^* = -0.158529474453882$ (private communication of the authors of [13]) for $N = 20$ and varying P are shown. The results indicate a quadratic order of convergence w.r.t. P .

P	$ \lambda^* - \lambda_{N,P} $
5	6.9856e-02
10	1.7990e-02
20	4.6015e-03
40	1.1568e-03
80	2.8960e-04
160	7.2425e-05

Table 1: Errors in the approximation of the real eigenvalue $\lambda^* = -0.158529474453882$ for $N = 20$

In the analysis of neural field equations Hopf bifurcations are relevant. Let us consider the model (24) with $\tau_0 = 1$, $\alpha = 1$, $\bar{c}_1 = 3$, $\bar{c}_2 = -5.5$, $\mu_1 = 0.5$, $\mu_2 = 1$ and assume that μ in the activation function is a parameter. By using the IG-approach with $N = 20$ and $P = 80$, we obtain that $\mu^* \approx 4.2202$ is Hopf bifurcation value (see Figure 2) in accordance with the results in [13, Table 1, Fig.3]. Finally we consider the connectivity kernel

$$W(x, y) = \bar{c}_1 e^{-\mu_1 |x-y-a|} + \bar{c}_2 e^{-\mu_2 |x-y+a|}, x, y \in \bar{\Omega}, \quad (28)$$

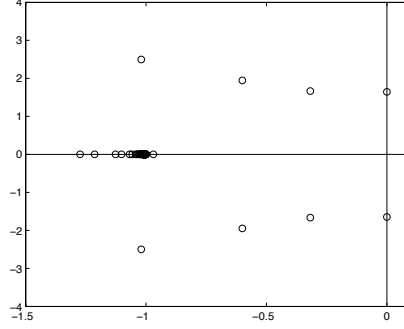


Figure 2: Rightmost part of the spectrum of $\mathcal{A}_{N,P}$ with $P = 80$ and $N = 20$ for $\mu = 4.2202$

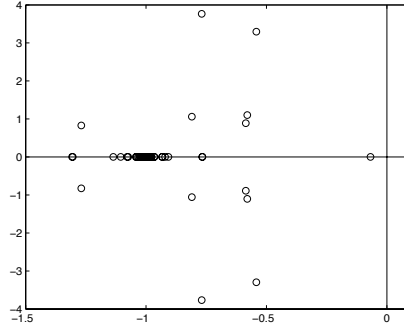


Figure 3: Rightmost part of the spectrum of $\mathcal{A}_{N,P}$ with $P = 80$, $N = 20$ and W in (28)

with $\bar{c}_1 = \bar{c}_1 = 2$, $\mu_1 = \mu_2 = 2$ and $a = 0.5$. For $\tau_0 = 1$ and $\mu = 4$ we obtain the results in Figure 3 and the zero equilibrium is locally asymptotically stable. Note the accumulation at the essential value $-\alpha = -1$.

5. Acknowledgements

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Figure

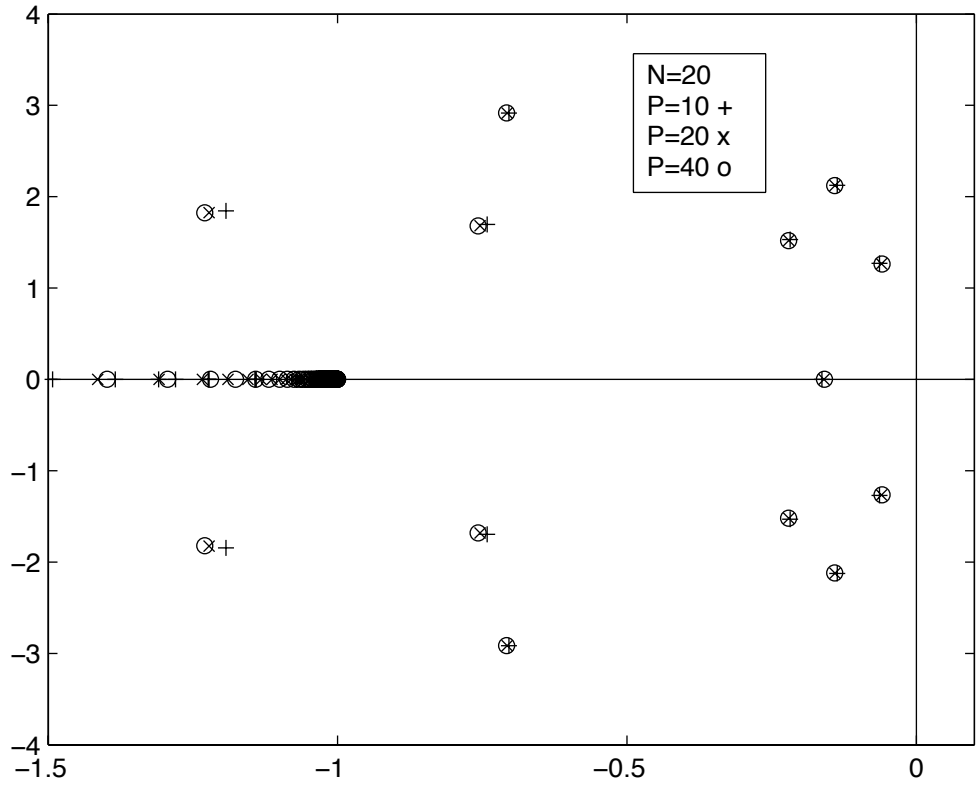


Figure 1

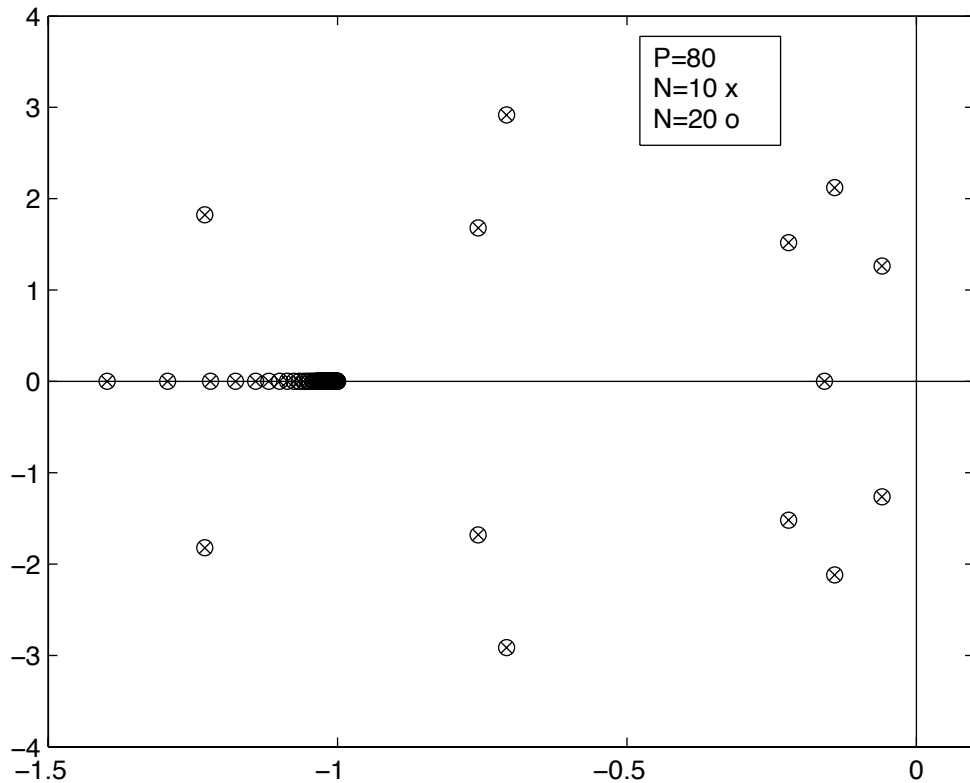


Figure 2

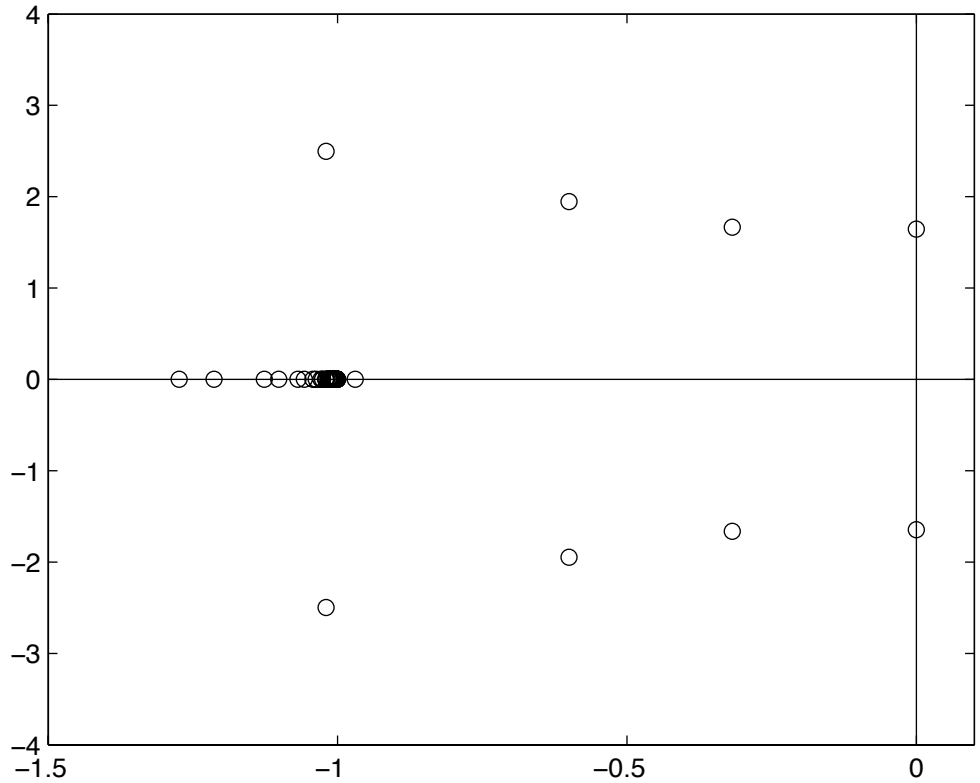


Figure 3

